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LIGHT OPEN AND OPEN MAPPINGS ON MANIFOLDS. II

BY

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ABSTRACT. Sufficient conditions are given for the existence of light open mappings between p.l. manifolds. In addition, it is shown that a mapping f from a p.l. manifold M^m, $m \geqslant 3$, to a polyhedron Q is homotopic to an open mapping of M onto Q iff the index of $f_*(\pi_1(M))$ in $\pi_1(Q)$ is finite. Finally, it is shown that an open mapping of M^m onto a p.l. manifold N^n, $n \geqslant m \geqslant 3$, can be approximated by a light open mapping of M onto N.

In [19], D. Wilson constructs examples of light open mappings (with each point inverse a Cantor set) from any 3 manifold onto any n cell ($n \geqslant 3$) and he constructs examples of monotone open mappings of any p.l. manifold M^m ($m \geqslant 3$) onto any n cell (these results answered questions raised by Eilenberg in [3]). In the first paper in this series [24], the author gave a complete analysis of the existence of monotone and monotone open mappings from manifolds onto polyhedra. In this paper, we give a complete analysis of the existence of open mappings from manifolds onto polyhedra (using results from [24] and from the theory of covering spaces); however, the principal content of this paper is the technique developed in §5 for constructing light open mappings between manifolds (with each point inverse a Cantor set). The techniques used in this paper are inspired by those of D. Wilson in [19] and [18]; indeed, the many similarities are apparent.

The "key" result which enables us to remove the assumption (of Wilson in [19]) that the domain manifold have dimension three is contained in the appendix (it is necessary to study §5 in order to understand the relevance of the appendix). The philosophy behind removing the assumption that the image is a cell is exactly the same as in [24] (however, we must assume the image is a manifold). In addition, the technical difficulties encountered in §5 are numerous.

In order to read this paper, it will be necessary to have a copy of the first part [24]. Indeed, we will need to refer to [24] so often that we have numbered the sections of this paper beginning with 4; *any reference to* §1, 2, *or* 3 *will be to that section of* [24] (eg., (3.7.1) refers to §3 of [24]). The notation used here is exactly as in [24], hence we shall not reproduce it.

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4. In this section, we develop an "up to homotopy" monotone open-light open factorization which leads to a proof of the following theorem.

(4.0) THEOREM. *A mapping from a compact, connected p.l. manifold M^m, $m \geqslant 3$, to a compact, connected polyhedron Q is homotopic to an open mapping of M onto Q if and only if the index of $f_*(\pi_1(M))$ in $\pi_1(Q)$ is finite.*

The "only if" half of this result is due to Smale [22]; we present in Proposition (4.1) a modified version of Smale's result which we prove using covering space theory. The reader is referrred to [23] for results on covering spaces used below. Recall that a mapping is *proper* provided the inverse image of each compact set is compact.

(4.1) PROPOSITION. *Let X and Y be connected metric spaces with Y semi-locally simply connected and let $f\colon (X, x_0) \longrightarrow (Y, y_0)$ be a proper, open mapping which is also onto. Then the index of $f_*(\pi_1(X, x_0))$ in $\pi_1(Y, y_0)$ is finite.*

PROOF. Let $p\colon (\widetilde{Y}, \widetilde{y}_0) \longrightarrow (Y, y_0)$ be a covering projection with $p_*(\pi_1(\widetilde{Y}, \widetilde{y}_0)) = f_*(\pi_1(X, x_0))$; recall that the index of $f_*(\pi_1(X, x_0))$ in $\pi_1(Y, y_0)$ is equal to the cardinality of $p^{-1}(y_0)$. Let $\widetilde{f}\colon (X, x_0) \longrightarrow (\widetilde{Y}, \widetilde{y}_0)$ be a lifting of f; that is, $p \circ \widetilde{f} = f$. It follows that $\widetilde{f}$ is proper and open and, hence, onto (recall that $\widetilde{Y}$ is connected). Since $\widetilde{f}^{-1}(p^{-1}(y_0)) = f^{-1}(y_0)$ is compact, it follows that $p^{-1}(y_0)$ is compact and, hence, finite.

PROOF OF THEOREM (4.0). The "if" half is proved as follows. Let $p\colon (\widetilde{Q}, \widetilde{q}_0) \longrightarrow (Q, q_0)$ be a covering projection with $p_*(\pi_1(\widetilde{Q}, \widetilde{q}_0)) = f_*(\pi_1(M, x_0))$. Since the index of $f_*(\pi_1(M, x_0))$ in $\pi_1(Q, q_0)$ is finite, we have that $p^{-1}(q_0)$ is finite; hence $\widetilde{Q}$ is a compact, connected polyhedron. Let $\widetilde{f}\colon (M, x_0) \longrightarrow (\widetilde{Q}, \widetilde{y}_0)$ be a lifting of f. It follows that $\widetilde{f}_*\colon \pi_1(M, x_0) \longrightarrow \pi_1(\widetilde{Q}, \widetilde{q}_0)$ is onto. Corollary (3.7.2) in [24] implies that $\widetilde{f}$ is homotopic to a monotone open mapping $\widetilde{g}$ from M onto $\widetilde{Q}$ (we do *not* claim that $\widetilde{g}$ preserves base points). Let $g = p \circ \widetilde{g}$; then g is homotopic to f and g is an open mapping of M onto Q. As promised earlier, we may view $p \circ \widetilde{g}$ as a monotone open-light open factorization of f (up to homotopy).

5. This section closely parallels §3 in [24]; Proposition (5.1) below contains the main technical tool for constructing light open mappings (as Proposition (3.1) did for constructing open mappings). Theorem (5.0) follows from Proposition (5.1) (and its proof) and Proposition 3 in [18]. Observe that the only difference between the hypothesis of the following theorem and that of Theorem (3.0) is that we assume that the range is a p.l. manifold of sufficient dimension, not simply a polyhedron.

(5.0) THEOREM. *Let M^m, N^n be compact, connected p.l. manifolds ($n \geq m \geq 3$) with triangulations K and L, respectively. Let P be a collection of nonempty subsets of M with pairwise disjoint interiors, with each $p \in P$ a union of elements of $t(K)$, and with $P^* = M$. Let T be a one to one function from P onto $t(L)$ satisfying*:

(5.0.1) $T(p_1) \cap \ldots \cap T(p_q) \neq \emptyset$ *whenever* $p_1 \cap \ldots \cap p_q \neq \emptyset$.

(5.0.2) *Each component of* $T^{-1}(\sigma_1)$ *meets* $T^{-1}(\sigma_2)$ *whenever* $\sigma_1 \cap \sigma_2 \neq \emptyset$, $\sigma_1, \sigma_2 \in t(L)$.

Then there is a light open mapping f from M onto N with $f^{-1}(y)$ homeomorphic to a Cantor set for each $y \in N$ and with $f(x) \in \mathrm{st}^2(T(p), L)^*$ for $x \in p \in P$.

It is best to view Proposition (5.1) as a (nontrivial) modification of Proposition (3.1). To this end, we will indicate how to modify the first part of the proof of (3.1); that is, up to but not including (3.5). At this point the two proofs differ radically and we will continue in detail with the remainder of the proof of (5.1).

(5.1) PROPOSITION. *Assume the hypothesis of Theorem* (5.0); *then there are two sequences of finite collections of polyhedra* $\{J_n\}_{n=1}^{\infty}$ *and* $\{K_n\}_{n=1}^{\infty}$ *satisfying*: (5.1.1), (5.1.2), . . . , (5.1.7) *are exactly the same as* (3.1.1), (3.1.2), . . . , (3.1.7).

(5.1.8) *If* $j_n^1 \cap \ldots \cap j_n^r \neq \emptyset$, *then the diameter of each component of* $R_n(j_n^1) \cup \ldots \cup R_n(j_n^r)$ *is less than* $28/2^{n-1}$.

PROOF OF (5.1). We shall use $(5.1.\cdot)^n$ to indicate condition $(5.1.\cdot)$ for the nth stage. The construction for $n = 1$ is done as in (3.1); we now proceed from the nth stage to the $(n+1)$st stage.

(5.2) Choose H and ϵ' as in (3.2) with the additional requirement that $\mathrm{st}^4(a, t(H))^* \cap \mathrm{st}^4(a', t(H))^* = \emptyset$ for $a \in A_{j_n}$, $a' \in A_{j'_n}$ and $a \neq a'$. We are now going to determine l_{n+1} and L_{n+1}; however, in doing so, we will also be setting up a considerable amount of machinery to be used later. The reader is advised to develop a schematic picture while going through the next paragraph; see Figure 1.

Let Q be an integer such that $\mathrm{ST}^Q(A_{j_n}^*, t(H|j_n))^* = R_n(j_n)$ for each $j_n \in J_n$. Choose l_{n+1} as follows. Recall that ϵ' is chosen as in (3.2); let $Q' = Q + 3m^2 + 5$ and let $l > l_n$ be such that the diameter of each $\mathrm{st}^{Q'+1}(\sigma, \beta^l L)^*$ is less than ϵ' for each $\sigma \in \beta^l L$ and such that $\mathrm{st}^{Q'}(j_n, \beta^l L)^* \subseteq \mathrm{st}(j_n, \beta^{l_n+1} L)^*$; this last condition will guarantee the conclusion in (5.0) that $f(x) \in \mathrm{st}^2(T(p), t(L))^*$. Let 3η = minimum of {distance of $N - \mathrm{st}^i(\sigma, \beta^l L)^*$ to $\mathrm{st}^{i-2}(\sigma, \beta^l L)^* \mid \sigma \in \beta^{l_n} L$ and $2 \leq i \leq Q'$}. Let $l' > l + 4$ be an integer such that the diameter

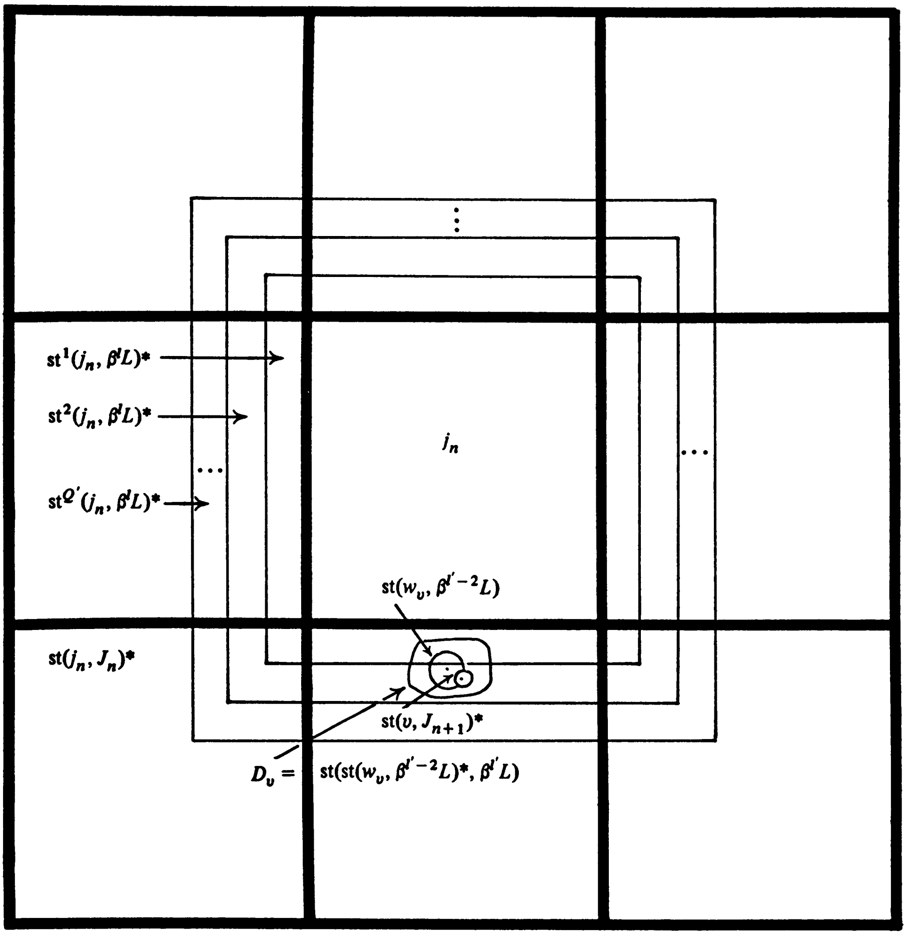

FIGURE 1

of $\mathrm{st}(\mathrm{st}(w, \beta^{l'-2}L)^*, \beta^{l'}L)^*$ is less than η for each vertex $w \in \beta^{l'-3}L$. Let $l_{n+1} > l'$ be an integer such that if v is a vertex of $\beta^{l_{n+1}}L$ and $v \in \mathrm{st}(w, \beta^{l'-2}L)^*$ for some vertex $w \in \beta^{l'-3}L$, then $\mathrm{st}(v, \beta^{l_{n+1}}L)^* \subseteq \mathrm{int}(\mathrm{st}(\mathrm{st}(w, \beta^{l'-2}L)^*, \beta^{l'}L)^*)$. Finally, let $L_{n+1} = Q'2^{l_{n+1}-l}$; the reader should check that $\mathrm{st}^{L_{n+1}}(j_n, J_{n+1})^* = \mathrm{st}^{Q'}(j_n, t(\beta^l L))^*$ and that if $j_{n+1} \in J_{n+1} = \beta^{l_{n+1}}L$ and $\sigma \in t(\beta^l L)$ with $j_{n+1} \subseteq \sigma$, then $\mathrm{st}^{L_{n+1}}(j_{n+1}, J_{n+1})^* \subseteq \mathrm{st}^{Q'}(\sigma, t(\beta^l L))^*$. In particular, the diameter of $\mathrm{st}^{L_{n+1}}(j_{n+1}, J_{n+1})^*$ is less than ϵ' and, therefore, condition $(5.1.2)^{n+1}$ holds.

(5.3) Read here exactly the rule stated in (3.3). As in (3.3), this basic rule guarantees that $(5.1.4)^{n+1}$ will hold and this rule together with $(5.1.7)^n$ (and the above choice of L_{n+1}) guarantees that $(5.1.5)^{n+1}$ and $(5.1.6)^{n+1}$ will hold. One fact deducible from the choices of Q, Q' and L_{n+1} is that if

$j_{n+1} \subseteq \mathrm{st}^{3m^2}(j_n, t(\beta^l L))^*$, then $R_{n+1}(j_{n+1})$ is to meet every member $h \in t(H)$ in $R_n(j_n)$.

(5.4) For each vertex v of J_{n+1} (more precisely, v is a vertex of $\beta^{l_{n+1}}L$), let $S(v) = \{\mathrm{bd}(\mathrm{st}(w, \beta^2 H)^*) | w$ a vertex of $\beta^1 H\}^*$; each $S(v)$ is a subcomplex of $\beta^2 H$. General position the $S(v)$'s with respect to each other and with respect to $\mathrm{cl}(H^{m-1} \cap (M - \partial M))$ so that the part of each $S(v)$ in ∂M (resp. $M - \partial M$) remains in ∂M (resp. $M - \partial M$). Let $S = \bigcup S(v)$ and let B_1 be a subdivision of H with each $S(v)$ a subcomplex of B_1.

Let $U = \mathrm{st}(S, \beta^2 B_1)^*$, let $K_n^U = \{k_n \cap \mathrm{cl}(M - U) | k_n \in K_n\}$, and let $R_n^U(j_n) = R_n(j_n) \cap \mathrm{cl}(M - U) \in K_n^U$.

(5.5) Most likely, the collection of components of the sets in K_n^U will not be simple (a fact we will have to live with). By using $l_{n+1} - l_n$ applications of the construction in (1.3) (begin with the triple $(R_n^U)^{-1}: K_n^U \longrightarrow J_n$ which satisfies (1.1.1), but omit the step where the collection of components is made simple), construct a triple $T_U: P_U \longrightarrow J_{n+1}$ satisfying:

(5.5.1) $P_U^* = (K_n^U)^*$ and $\mathrm{cl}(\mathrm{int}(p)) = p$ for each $p \in P_U$; the collection of components of the sets in P_U may *not* be simple.

(5.5.2) If $j_{n+1} \subseteq j_n$, then $T_U^{-1}(j_{n+1}) \subseteq R_n^U(j_n)$.

(5.5.3) $T_U(p_1) \cap \ldots \cap T_U(p_q) \neq \varnothing$ whenever $p_1 \cap \ldots \cap p_q \neq \varnothing$.

(5.5.4) If C, C' are components of $\mathrm{cl}(h - U)$, $\mathrm{cl}(h' - U)$, $h \subseteq R_n^U(j_n)$ and $h' \subseteq R_n^U(j'_n)$, and $C \cap C'$ contains an $m - 1$ cell, then for each pair $j_{n+1} \subseteq j_n$, $j'_{n+1} \subseteq j'_n$ with $j_{n+1} \cap j'_{n+1} \neq \varnothing$, $\mathrm{int}(T_U^{-1}(j_{n+1}) \cup T_U^{-1}(j'_{n+1})) \cap \mathrm{int}(C \cup C')$ is connected (possibly, $C = C'$, $j_n = j'_n$, or $j_{n+1} = j'_{n+1}$).

By running additional tubes during each application of (1.3), we can assume in addition:

(5.5.5) If C is a component of $\mathrm{cl}(h - U)$, $h \subseteq R_n(j_n)$, and $\sigma \subseteq C \cap U$, σ an $m - 1$ simplex of $\beta^2 B_1$, then $T_U^{-1}(j_{n+1}) \cap \sigma$ contains an $m - 1$ cell for each $j_{n+1} \subseteq j_n$.

(5.6) We are now going to alter the triple $T_U: P_U \longrightarrow J_{n+1}$ so that $T_U^{-1}(j_{n+1})$ meets each set in $t(H)$ which $R_{n+1}(j_{n+1})$ is supposed to meet (see (5.3)).

Let C be a component of $\mathrm{cl}(h - U)$, $h \subseteq R_n(j_n)$ and $h \in t(H)$; let r be an integer such that $R_{n+1}(j_{n+1})$ is to meet h if and only if $j_{n+1} \in \mathrm{st}^r(j_n, J_{n+1})$. For each $j_{n+1} \in \mathrm{st}^1(j_n, J_{n+1}) - \mathrm{st}^0(j_n, J_{n+1})$ alter $T_U^{-1}(j_{n+1})$ by adding to it an m cell in $\mathrm{int}(T_U^{-1}(j'_{n+1})) \cap \mathrm{int}(C)$ where $j_{n+1} \cap j'_{n+1} \neq \varnothing$ and $j'_{n+1} \in \mathrm{st}^0(j_n, J_{n+1})$; remove the interior of the m cell from $T_U^{-1}(j'_{n+1})$. Inductively, for $i = 2, \ldots, r$, if $j_{n+1} \in \mathrm{st}^i(j_n, J_{n+1}) - \mathrm{st}^{i-1}(j_n, J_{n+1})$, then add to $T_U^{-1}(j_{n+1})$ an m cell in $\mathrm{int}(T_U^{-1}(j'_{n+1})) \cap \mathrm{int}(C)$ where $j_{n+1} \cap j'_{n+1'} \neq \varnothing$ and $j'_{n+1} \in \mathrm{st}^{i-1}(j_n, J_{n+1})$; remove the interior of the m cell from $T_U^{-1}(j'_{n+1})$.

Let $P_U^C = \{\mathrm{cl}(T_U^{-1}(j_{n+1}) \cap \mathrm{int}(C)) | j_{n+1} \in \mathrm{st}^r(j_n, J_{n+1})\}$ and let $T_U^C(\mathrm{cl}(T_U^{-1}(j_{n+1}) \cap \mathrm{int}(C))) = j_{n+1}$ for $j_{n+1} \in \mathrm{st}^r(j_n, J_{n+1})$. Apply the techniques in (2.5) – (2.8) to the triple $T_U^C\colon P_U^C \longrightarrow \mathrm{st}^r(j_n, J_{n+1})$, using the fact that $\mathrm{st}^r(j_n, J_{n+1})^*$ is simple connected, to alter the sets $T_U^{-1}(j_{n+1}), j_{n+1} \in \mathrm{st}^r(j_n, J_{n+1})$ so that:

(5.6.1) If $j_{n+1}, j'_{n+1} \in \mathrm{st}^r(j_n, J_{n+1})$ with $j_{n+1} \cap j'_{n+1} \neq \varnothing$, then $\mathrm{int}(T_U^{-1}(j_{n+1}) \cup T_U^{-1}(j'_{n+1})) \cap \mathrm{int}(C)$ is connected.

(5.7) After the alterations of (5.6) have been made for each $h \in t(H)$ and each component C of $\mathrm{cl}(h - U)$, the altered triple $T_U\colon P_U \longrightarrow J_{n+1}$ will still satisfy (5.5.1), (5.5.3), (5.5.4), and (5.6.1); be warned that the superscript "r" in condition (5.6.1) depends on the set h for which we have C a component of $\mathrm{cl}(h - U)$. In addition, $T_U^{-1}(j_{n+1})$ meets every member of $t(H)$ which $R_{n+1}(j_{n+1})$ is to meet and $T_U^{-1}(j_{n+1}) \cap H^{m-1} \subseteq R_n(j_n)$ where $j_{n+1} \subseteq j_n$.

Assessing what we have done so far, we observe that the triple $T_U\colon P_U \longrightarrow J_{n+1}$ satisfies all the conditions $(5.1.\cdot)^{n+1}$ (with T_U^{-1} and P_U in place of R_{n+1} and K_{n+1}) except for $(5.1.1)^{n+1}$ and $(5.1.7)^{n+1}$. The two major problems we face are that $P_U^* \neq M$ and that each component of $T_U^{-1}(j_{n+1})$ may not meet $T_U^{-1}(j'_{n+1})$ even though $j'_{n+1} \cap j_{n+1} \neq \varnothing$. §§(5.8) – (5.11) deal with the latter problem and (5.12) the former.

Before proceeding, let us discuss the role of the $S(v)$'s. Ideally, we would like $R_{n+1}(j_{n+1}) \cap S(v) = \varnothing$ for each $v \in j_{n+1}$ as this would immediately give us $(5.1.8)^{n+1}$. We will *not* have this ideal situation, but we will "control" this intersection so that we can still deduce $(5.1.8)^{n+1}$.

(5.8) Let v be a vertex of J_{n+1} and let $h \in t(H)$ where $h \subseteq R_n(j_n)$. For each pair D, D' of components of $h - S$ with $\mathrm{cl}(D) \cap \mathrm{cl}(D')$ containing an $m-1$ cell in $S(v)$ make the following alterations. (Let C, C' be the unique components of $h - U$ with $C \subseteq D$, $C' \subseteq D'$.) For each $j_{n+1} \subseteq j_n$ with $j_{n+1} \cap N_\eta(v) = \varnothing$ (recall that η was defined in (5.2)), use condition (5.5.5) to connect $T_U^{-1}(j_{n+1}) \cap \mathrm{cl}(C)$ to $T_U^{-1}(j_{n+1}) \cap \mathrm{cl}(C')$ with a tube in $T_U^{-1}(j_{n+1}) \cup (D - C) \cup (D' - C) \cup S(v)$. Note that this tube does not meet $S - S(v)$ and be sure that no two such tubes intersect. Add this tube to $T_U^{-1}(j_{n+1})$. Make the above alterations for each vertex v of J_{n+1} and each $h \in t(H)$. Observe that we *no longer have* $P_U^* = \mathrm{cl}(M - U)$.

(5.9) We will now make the necessary alterations so that if $j_{n+1} \cap j'_{n+1} \neq \varnothing$, then each component of $T_U^{-1}(j_{n+1})$ will meet $T_U^{-1}(j'_{n+1})$. Let us assume that the $S(v)$'s were general positioned in (5.4) so that for each pair $h, h' \in t(H)$ with $h \cap h'$ an $m-1$ simplex, there are components C_h and $C_{h'}$ of $h - U$ and $h' - U$ such that $C_h \cap C_{h'}$ contains an $m-1$ cell and such that for each vertex v' of J_{n+1} the closure of the component D_h (resp., $D_{h'}$) of $h - S(v')$ (resp., $h' - S(v')$) containing C_h (resp., $C_{h'}$) does not intersect $\mathrm{bd}(h) - h \cap h'$ (resp.,

$\mathrm{bd}(h') - h \cap h'$). Choosing such a "general positioning" is not difficult (in fact, almost any "natural" choice will do).

For each $j_n \in J_n$, let $\widetilde{t}(H|j_n) = \{h \in t(H|j_n) | h \not\subseteq A^*_{j_n}\}$. For each $h \in \widetilde{t}(H|j_n)$, let r_h be the unique integer with

$$h \in \mathrm{ST}^{L_{n+1}-r_h}(A^*_{j_n}, t(H|j_n)) - \mathrm{ST}^{L_{n+1}-r_h-1}(A^*_{j_n}, t(H|j_n)).$$

It follows from the choices of Q', Q and L_{n+1} in (5.2) that $r_h \geq 3m^2 + 1$.

For each $h \in \widetilde{t}(H|j_n)$, let $\sigma(h) \in \mathrm{ST}^{L_{n+1}-r_h-1}(A^*_{j_n}, t(H|j_n))$ with $h \cap \sigma(h)$ an $m-1$ simplex and let C_h and $C_{\sigma(h)}$ be the components of $h - U$ and $\sigma(h) - U$ discussed above.

For each $a \in A_{j_n}$, define a collection $\Sigma(a)$ as follows. For each $j'_n \in \mathrm{st}(j_n, J_n)$ with $j'_n \neq j_n$, let $h'_{j'_n}, h_{j'_n} \in t(H)$ with $h'_{j'_n} \subseteq a$, $h_{j'_n} \subseteq R_n(j'_n)$, and $h'_{j'_n} \cap h_{j'_n}$ an $m-1$ simplex; let $C_{h'_{j'_n}}$ and $C_{h_{j'_n}}$ be components of $h'_{j'_n} - U$ and $h_{j'_n} - U$ discussed above. Let

$$\Sigma(a) = \{(C_{h'_{j'_n}}, C_{h_{j'_n}}) | j'_n \in \mathrm{st}(j_n, J_n) \text{ and } j'_n \neq j_n\};$$

we emphasize that $\Sigma(a)$ contains *exactly one pair* $(C_{h'_{j'_n}}, C_{h_{j'_n}})$ for each $j'_n \in \mathrm{st}(j_n, J_n)$ with $j'_n \neq j_n$.

Let

$$\Gamma_{j_n} = \{(h, j_{n+1}) | h \in \widetilde{t}(H|j_n) \text{ and } j_{n+1} \in \mathrm{st}^{r_h+1}(j_n, J_{n+1}) - \mathrm{st}^{r_h}(j_n, J_{n+1})\},$$

$$\Omega_{j_n} = \{(a, j_{n+1}) | a \in A_{j_n} \text{ and } j_{n+1} \in \mathrm{st}^{L_{n+1}+1}(j_n, J_{n+1}) - \mathrm{st}^{L_{n+1}}(j_n, J_{n+1})\}.$$

For each $(h, j_{n+1}) \in \Gamma_{j_n}$ (resp., $(a, j_{n+1}) \in \Omega_{j_n}$) we will make $T_U^{-1}(j_{n+1}) \cap \mathrm{int}(h)$ (resp., $T_U^{-1}(j_{n+1}) \cap \mathrm{int}(a)$) connected and then we will connect $T_U^{-1}(j_{n+1}) \cap C_h$ to $T_U^{-1}(j_{n+1}) \cap C_{\sigma(h)}$ (resp., $T_U^{-1}(j_{n+1}) \cap C_{h'_{j'_n}}$ to $T_U^{-1}(j_{n+1}) \cap C_{h_{j'_n}}$ where $j_{n+1} \subseteq j'_n$ and $(C_{h'_{j'_n}}, C_{h_{j'_n}}) \in \Sigma(a)$). We emphasize that the same $\sigma(h)$ (resp., $\Sigma(a)$) is used for each pair (h, j_{n+1}) (resp., (a, j_{n+1})). The problem we need to remedy is that for $(h, j_{n+1}) \in \Gamma_{j_n}$ (resp., $(a, j_{n+1}) \in \Omega_{j_n}$), if $j'_{n+1} \in \mathrm{st}^{r_h+2}(j_n, J_{n+1}) - \mathrm{st}^{r_h+1}(j_n, J_{n+1})$ (resp., $j'_{n+1} \in \mathrm{st}^{L_{n+1}+2}(j_n, J_{n+1}) - \mathrm{st}^{L_{n+1}+1}(j_n, J_{n+1})$) with $j'_{n+1} \cap j_{n+1} \neq \varnothing$, then $T_U^{-1}(j'_{n+1}) \cap h = \varnothing$ (resp., $T_U^{-1}(j'_{n+1}) \cap a = \varnothing$); hence, the components of $T_U^{-1}(j_{n+1})$ in h (resp., a) do not meet $T_U^{-1}(j_{n+1})$. However, since $T_U^{-1}(j'_{n+1}) \cap \sigma(h) \neq \varnothing$ (resp., $T_U^{-1}(j'_{n+1}) \cap h_{j'_n} \neq \varnothing$ where $j'_{n+1} \subseteq j'_n$), once we have made the changes indicated above, we will have eliminated this problem. The changes we now describe are to be made for each $j_n \in J_n$.

For each $(h, j_{n+1}) \in \Gamma_{j_n}$ make the following alterations. Let D and D' be components of $h - S$ with $\mathrm{cl}(D) \cap \mathrm{cl}(D')$ containing an $m-1$ cell; let v be such that $\mathrm{cl}(D) \cap \mathrm{cl}(D')$ contains an $m-1$ cell in $S(v)$. Let $j^1_{n+1}, \ldots, j^t_{n+1} =$

j_{n+1} be a chain (i.e., $j_{n+1}^i \cap j_{n+1}^{i+1} \neq \emptyset$, $i = 1, \ldots, t-1$) in $\mathrm{st}^{r_h+1}(j_n, J_{n+1})$ with $j_{n+1}^1 \subseteq j_n$, with $v \notin j_{n+1}^i$ for $i = t-2, \ldots, 1$, and with $j_{n+1}^1 \cap N_\eta(v) = \emptyset$. In view of the changes made in (5.8), $T_U^{-1}(j_{n+1}^1) \cap (\mathrm{int}(D \cup D' \cup S(v)))$ is connected; connect $T_U^{-1}(j_{n+1}^2) \cap D$ to $T_U^{-1}(j_{n+1}^2) \cap D'$ with a tube in $\mathrm{int}(T_U^{-1}(j_{n+1}^2) \cup T_U^{-1}(j_{n+1}^1)) \cap (D \cup D' \cup S(v))$. The "new" $T_U^{-1}(j_{n+1}^2)$ meets $\mathrm{int}(D \cup D' \cup S(v))$ in a connected set; hence, we can connect $T_U^{-1}(j_{n+1}^3) \cap D$ to $T_U^{-1}(j_{n+1}^3) \cap D'$ with a tube in $\mathrm{int}(T_U^{-1}(j_{n+1}^3) \cup T_U^{-1}(j_{n+1}^2)) \cap (D \cup D' \cup S(v))$. In this manner, successively for $i = 4, \ldots, t$, connect $T_U^{-1}(j_{n+1}^i) \cap D$ to $T_U^{-1}(j_{n+1}^i) \cap D'$ with a tube in $\mathrm{int}(T_U^{-1}(j_{n+1}^i) \cup T_U^{-1}(j_{n+1}^{i-1})) \cap (D \cup D' \cup S(v))$. Make the above alteration for each pair D, D' of components of $h - S$ with $\mathrm{cl}(D) \cap \mathrm{cl}(D')$ containing an $m-1$ cell. Let $j_{n+1}^1, \ldots, j_{n+1}^t = j_{n+1}$ be a chain in $\mathrm{st}^{r_h+1}(j_n, J_{n+1})$ with $j_{n+1}^1 \subseteq j_n$. In view of (5.5.4), $T_U^{-1}(j_{n+1}^1) \cap \mathrm{int}(C_h \cup C_{\sigma(h)})$ is connected. As above, successively for $i = 2, \ldots, t$, connect $T_U^{-1}(j_{n+1}^i) \cap C_h$ to $T_U^{-1}(j_{n+1}^i) \cap C_{\sigma(h)}$ with a tube in

$$\mathrm{int}(T_U^{-1}(j_{n+1}^i) \cup T_U^{-1}(j_{n+1}^{i-1})) \cap \mathrm{int}(C_h \cup C_{\sigma(h)}).$$

For each $(a, j_{n+1}) \in \Omega_{j_n}$ make the following alterations. Using the method in the preceding paragraph, make $T_U^{-1}(j_{n+1}) \cap \mathrm{int}(a)$ connected by connecting $T_U^{-1}(j_{n+1}) \cap D$ to $T_U^{-1}(j_{n+1}) \cap D'$ for each pair of components D, D' of $a - (S \cup H^{m-1})$ with $\mathrm{cl}(D) \cap \mathrm{cl}(D')$ containing an $m-1$ cell. (If connecting across an $m-1$ cell contained in H^{m-1}, then any chain $j_{n+1}^1, \ldots, j_{n+1}^t = j_{n+1}$ in $\mathrm{st}^{L_{n+1}+1}(j_n, J_{n+1})$ can be used; however, if the $m-1$ cell is contained in $S(v)$, then choose the chain so that $v \notin j_{n+1}^i$ for $i = t-2, \ldots, 1$ and so that $j_{n+1}^1 \cap N_\eta(v) = \emptyset$.) Finally, let j_n' be such that $j_{n+1} \subseteq j_n'$ and let $j_{n+1}^1, \ldots, j_{n+1}^t = j_{n+1}$ be a chain in $\mathrm{st}^{L_{n+1}+1}(j_n, J_{n+1})$ with $j_{n+1}^1 \subseteq j_n$ and $j_n^i \subseteq j_n'$ for $i = 2, \ldots, t$. In view of (5.5.4)

$$\mathrm{int}(T_U^{-1}(j_{n+1}^2) \cup T_U^{-1}(j_{n+1}^1)) \cap \mathrm{int}(C_{h_{j_n'}} \cup C_{h_{j_n'}})$$

is connected; hence, we can connect $T_U^{-1}(j_{n+1}^2) \cap C_{h_{j_n'}}$ to $T_U^{-1}(j_{n+1}^2) \cap C_{h_{j_n'}}$ with a tube in this set. Now, successively for $i = 3, \ldots, t$, connect $T_U^{-1}(j_{n+1}^i) \cap C_{h_{j_n'}}$ to $T_U^{-1}(j_{n+1}^i) \cap C_{h_{j_n'}}$ with a tube in

$$\mathrm{int}(T_U^{-1}(j_{n+1}^i) \cup T_U^{-1}(j_{n+1}^{i-1})) \cap (C_{h_{j_n'}} \cup C_{h_{j_n'}}).$$

(5.10) We now have that if $j_{n+1} \neq \emptyset$, then the intersection of each component of $T_U^{-1}(j_{n+1})$ with $T_U^{-1}(j_{n+1}')$ contains an $m-1$ cell; and, in place of having $T_U^{-1}(j_{n+1}) \cap S(v) = \emptyset$ for each $v \in j_{n+1}$, we have that:

(5.10.1) For each vertex v of J_{n+1}, each component of $\{T_U^{-1}(j_{n+1}) | v \in j_{n+1}\}^*$ is a subset of $\mathrm{st}^3(h, t(H))^*$ or $\mathrm{st}^3(a, t(H))^*$ for some $h \in t(H)$ or $a \in \bigcup A_{j_n}$.

We will now outline a proof of the validity of (5.10.1). Let B be a component of $\{T_U^{-1}(j_{n+1})|v \in j_{n+1}\}^*$ and suppose that $B \cap S(v) \neq \emptyset$ (if $B \cap S(v) = \emptyset$, then we are done). Let $s = \min\{s' |$ for some $j_n \in J_n$ and for some $h \in \mathrm{ST}^{s'}(A^*_{j_n}, t(H|j_n))$ we have that $B \cap S(v) \cap h \neq \emptyset\}$. Let us assume that $s \neq 0$ (the case $s = 0$ is handled similarly; in particular, using the fact that $\mathrm{st}^4(a, t(H))^* \cap \mathrm{st}^4(a', t(H))^* = \emptyset$, $a \neq a'$, one shows that if $B \cap S(v) \cap a \neq \emptyset$, then $B \subseteq \mathrm{st}^3(a, t(H))^*$).

Let $j_n \in J_n$ be such that there is

$$h \in \mathrm{ST}^s(A^*_{j_n}, t(H|j_n)) - \mathrm{ST}^{s-1}(A^*_{j_n}, t(H|j_n))$$

with $B \cap S(v) \cap h \neq \emptyset$ (it will turn out that both j_n and h are unique; this is not obvious). Recalling the definition of r_h, it certainly is the case that $s = L_{n+1} - r_h$. One of the "controls" governing the changes made in §(5.9) involved limiting the possible intersection of $T_U^{-1}(j_{n+1})$ with $S(v)$ for each $j_{n+1} \in \mathrm{st}(v, J_{n+1})$ (recall the carefully chosen chains used in the final two paragraphs of (5.9)); in particular, one can verify that we must have that $v \in \mathrm{st}^q(j_n, J_{n+1})^* - \mathrm{st}^{q-1}(j_n, J_{n+1})^*$ where q is equal to one of the following: $r_h + 1, r_h$, or $r_h - 1$. Let us assume that $q = r_h$ (the remaining two cases can be handled similarly). Let $h_1, \ldots, h_t$ be all the sets in $\mathrm{ST}^{L_{n+1}-r_h+1}(A^*_{j_n}, t(H|j_n)) - \mathrm{ST}^{L_{n+1}-r_h}(A^*_{j_n}, t(H|j_n))$ for which $\sigma(h_i) = h$, $i = 1, \ldots, t$. Letting $G = h \cup \sigma(h) \cup (\bigcup_{i=1}^t h_i)$, we claim that $B \subseteq \mathrm{int}(G)$; this fact can be deduced as follows. Let $j_{n+1} \in \mathrm{st}(v, J_{n+1})$; since $v \in \mathrm{st}^{r_h}(j_n, J_{n+1})^* - \mathrm{st}^{r_h-1}(j_n, J_{n+1})^*$, we must have that

$$j_{n+1} \in \mathrm{st}^{r_h}(j_n, J_{n+1}) - \mathrm{st}^{r_h-1}(j_n, J_{n+1}) \quad \text{or}$$

$$j_{n+1} \in \mathrm{st}^{r_h+1}(j_n, J_{n+1}) - \mathrm{st}^{r_h}(j_n, J_{n+1}).$$

We need to know where $T_U^{-1}(j_{n+1})$ intersects $\mathrm{bd}(G)$. (Before the changes in (5.9) were made, we had that $T_U^{-1}(j_{n+1}) \cap H^{m-1} \subseteq R_n(j'_n)$ where $j_{n+1} \subseteq j'_n$; since $G \cap R_n(j'_n) = \emptyset$, $T_U^{-1}(j_{n+1}) \cap \mathrm{bd}(G)$ is completely determined by the changes made in (5.9).) Letting $C_1 = \{C_{h'} \cap C_{\sigma(h)} | h' \in \mathrm{ST}^s(A^*_{j_n}, t(H|j_n))$ with $h' \neq h$ and $\sigma(h') = \sigma(h)\}^*$ and $C_2 = \{C_{h'} \cap C_{h_i} | h' \in \mathrm{ST}^{s+2}(A^*_{j_n}, t(H|j_n))$ with $\sigma(h') = h_i$ for some $i = 1, \ldots, t\}^*$, certainly, we have that $T_U^{-1}(j_{n+1}) \cap \mathrm{bd}(G) \subseteq C_1 \cup C_2$. However, the choice of minimal s implies that $B \cap S(v) \cap \sigma(h) = \emptyset$; therefore, because of the choice of component $C_{\sigma(h)}$ in (5.9), we have that $B \cap C_1 = \emptyset$. In addition, if $\sigma(h') = h_i$ for some $i = 1, \ldots, t$, then $h' \in \mathrm{ST}^{s+2}(A^*_{j_n}, t(H|j_n)) - \mathrm{ST}^{s+1}(A^*_{j_n}, t(H|j_n))$ and $T_U^{-1}(j_{n+1}) \cap h' = \emptyset$; therefore $B \cap C_2 = \emptyset$. Hence, we have that $B \subseteq \mathrm{int}(G)$.

In dealing with the case that $q = r_h - 1$, in place of the set G, it is necessary to use the set

$$G' = \left(h \cup \sigma(h) \cup \left(\bigcup_{i=1}^{t} h_i\right)\right)$$

$$\cup \{h' | h' \in \mathrm{ST}^{s+2}(A^*_{j_n}, t(H|j_n)) \text{ with } \sigma(h') = h_i \text{ for some } i = 1, \ldots, t\}^*.$$

If $q = r_h + 1$, then the set G suffices.

(5.11) We are now in a position to construct a collection of m cells $A'_{j_{n+1}}$, for each $j_{n+1} \in J_{n+1}$, satisfying condition $(5.1.7)^{n+1}$ for the triple T_U: $P_U \longrightarrow J_{n+1}$. For each $j_{n+1} \in J_{n+1}$ and for each component W of $T_U^{-1}(j_{n+1})$ do the following. Let $h \in t(H)$ with $W \cap T_U^{-1}(j'_{n+1}) \cap \mathrm{int}(h)$ containing an $m - 1$ cell for each $j'_{n+1} \in \mathrm{st}(j_{n+1}, J_{n+1})$; for certain W such an h existed before the changes in (5.9) were made and for the others the changes in (5.9) yield such an h. Choose an m cell in $\mathrm{int}(W) \cap \mathrm{int}(h)$ of diameter less than $4/2^{n+1}$ and, for each $j'_{n+1} \in \mathrm{st}(j_{n+1}, J_{n+1})$, connect $T_U^{-1}(j'_{n+1}) \cap h$ to this m cell with a tube in $\mathrm{int}(W \cup T_U^{-1}(j'_{n+1})) \cap \mathrm{int}(h)$; this m cell is to be an element of $A'_{j_{n+1}}$.

(5.12) Let $V = \mathrm{cl}(U - \{T_U^{-1}(j_{n+1}) | j_{n+1} \in J_{n+1}\}^*)$ and let B_2 be a subdivision of B_1 which subdivides V. In this section we will construct a (particular) p.l. mapping from V to N and in (5.13) we will use this mapping to "enlarge" the $T_U^{-1}(j_{n+1})$'s so that they "fill" all of M.

Using the *machinery set up in* (5.2) and Proposition A from the appendix, we now construct a p.l. mapping g: $V \longrightarrow N$ satisfying:

(5.12.1) $g(S(v) \cap V) \cap \mathrm{st}(v, J_{n+1})^* = \emptyset$ for each vertex v of J_{n+1}.

(5.12.2) If $\sigma \in B_2$ and $\sigma \subseteq \mathrm{bd}(V)$, then $g(\sigma) \subseteq \bigcap \{j_{n+1} | \sigma \subseteq T_U^{-1}(j_{n+1})\}$.

(5.12.3) $g(V \cap R_n(j_n)) \subseteq \mathrm{st}^{3m^2}(j_n, \beta^l L)^*$ for each $j_n \in J_n$.

For each vertex $v \in J_{n+1}$, let w_v be a vertex of $\beta^{l'-3}L$ with $v \in \mathrm{st}(w_v, \beta^{l'-2}L)^*$ and let $D_v = \mathrm{st}(\mathrm{st}(w_v, \beta^{l'-2}L)^*, \beta^{l'}L)^*$; note that $\mathrm{st}(v, J_{n+1})^* \subseteq \mathrm{int}(D_v)$; see Figure 1. Define g on $\mathrm{bd}(V)$ inductively, for $i = 0, \ldots, m - 1$, by mapping each i simplex $\sigma \in B_2$ into $\bigcap \{j_{n+1} | \sigma \subseteq T_U^{-1}(j_{n+1})\}$ (see (1.1) for more details). Observe that the set $S \cap \mathrm{bd}(V)$ is completely determined by the changes made in (5.8); in particular, if $\sigma \subseteq S \cap \mathrm{bd}(V)$, then there is a unique vertex v of J_{n+1} with $\sigma \subseteq S(v) \cap \mathrm{bd}(V)$. Furthermore, we have that $g(\sigma) \cap \mathrm{int}(D_v) = \emptyset$ (since the diameter of D_v is less than η, $D_v \subseteq N_\eta(v)$; now recall the condition in the fourth sentence of §(5.8)).

If y is a vertex of B_2 in $S \cap V - \mathrm{bd}(V)$, then let $v_1, \ldots, v_s$ be all the vertices of J_{n+1} with $y \in S(v_i)$, $i = 1, \ldots, s$; because of the general positioning done in (5.4), we have that $s \leqslant m$. Using the machinery set up in (5.2), we can find an integer r, $0 < r \leqslant 3m$, such that $D_{v_i} \cap \mathrm{bd}(\mathrm{st}^r(\bigcap \{j_n | y \in R_n(j_n)\}, \beta^l L)^*) = \emptyset$ for each $i = 1, \ldots, s$; in general, the D_{v_i}'s will be located throughout M; we are concerned with avoiding those D_{v_i}'s contained in

$$\mathrm{st}^{3m}\left(\bigcap\{j_n | y \in R_n(j_n)\}, \beta^l L\right)^*.$$

Let $g(y) \in \mathrm{int}(\mathrm{st}^r(\bigcap\{j_n | y \in R_n(j_n)\}, \beta^l L)^*)$ but $g(y) \notin \mathrm{int}(\bigcup_{i=1}^s D_{v_i})$.

If τ is a 1 simplex of B_2 in $S \cap V$ $(\tau \not\subseteq \mathrm{bd}(V))$, then let $v_1, \ldots, v_s$ be all the vertices of J_{n+1} with $\tau \subseteq S(v_i)$, $i = 1, \ldots, s$ (we have that $s \leqslant m - 1$). Let r be an integer, $3m < r \leqslant 2(3m)$, with

$$D_{v_i} \cap \mathrm{bd}\left(\mathrm{st}^r\left(\bigcap\{j_n | \tau \subseteq R_n(j_n)\}, \beta^l L\right)^*\right) = \varnothing$$

for each $i = 1, \ldots, s$. Extend g to τ by mapping τ into

$$\mathrm{int}\left(\mathrm{st}^r\left(\bigcap\{j_n | \tau \subseteq R_n(j_n)\}, \beta^l L\right)^*\right)$$

but with $g(\tau) \cap \mathrm{int}(\bigcup_{i=1}^s D_{v_i}) = \varnothing$ by using Proposition A (see the Appendix) with $q = l'$ and $B = \mathrm{st}^r(\bigcap\{j_n | \tau \subseteq R_n(j_n)\}, \beta^l L)^*$ (ignore those D_{v_i}'s not contained in B).

In general, to extend g from the q skeleton of $S \cap V$ to the $q + 1$ skeleton of $S \cap V$, let τ be a $q + 1$ simplex in $S \cap V$ $(\tau \not\subseteq \mathrm{bd}(V))$ and let $v_1, \ldots, v_s$ be all the vertices of J_{n+1} with $\tau \subseteq S(v_i)$, $i = 1, \ldots, s$ (we have that $s \leqslant m - q - 1$). Let r be an integer, $q(3m) < r \leqslant (q + 1)3m$ ... (continue reading from the third line of the preceding paragraph).

Extend g to all of V by mapping a q simplex $\tau \subseteq V$ $(\tau \not\subseteq S)$ into $\mathrm{int}(\mathrm{st}^{3m^2}(\bigcap\{j_n | \tau \subseteq R_n(j_n)\}, \beta^l L)^*)$; more precisely, do this inductively beginning with the vertices of B_2 in $V - S$.

(5.13) Let B_3 be a subdivision of B_2 such that g maps each simplex of B_3 contained in V linearly into a simplex of J_{n+1}.

Let $\tau \in t(B_3)$ with $\tau \not\subseteq P_U^*$ and with $\tau \cap P_U^* \cap \mathrm{int}(h)$ containing an $m - 1$ cell where $h \in t(H)$ is such that $\tau \subseteq h$. Let D be the component of $h - S$ with $\tau \subseteq \mathrm{cl}(D)$. Let $j_{n+1} \in J_{n+1}$ be such that $\tau \cap T_U^{-1}(j_{n+1}) \cap \mathrm{int}(h)$ contains an $m - 1$ cell; property (5.12.2) implies that if $j'_{n+1} \in J_{n+1}$ is such that $g(\tau) \subseteq j'_{n+1}$, then $j_{n+1} \cap j'_{n+1} \neq \varnothing$. Properties (5.12.3) and (5.6.1) and the comment in the last sentence of (5.3) guarantee that we can run a tube in

$$\mathrm{int}(T_U^{-1}(j_{n+1}) \cup \tau) \cap D$$

connecting τ to $T_U^{-1}(j'_{n+1})$; add τ and this tube to $T_U^{-1}(j'_{n+1})$ and replace $T_U^{-1}(j_{n+1})$ by $\mathrm{cl}(T_U^{-1}(j_{n+1})\text{-tube})$. Let $T_{U,1}: P_{U,1} \longrightarrow J_{n+1}$ denote this new triple.

Repeat the alterations of the preceding paragraph using $T_{U,1}: P_{U,1} \longrightarrow J_{n+1}$ in place of $T_U: P_U \longrightarrow J_{n+1}$ subject to the following modifications. Choose $\tau \in t(B_3)$ so that $\tau \cap P_{U,1}^* \cap \mathrm{int}(h)$ contains an $m - 1$ cell *not in* S; and if $\tau \cap P_{U,1}^* \cap \mathrm{int}(h)$ is not contained in $\mathrm{bd}(V)$, then read "g being a continuous function" in place of "property (5.12.2)". Call the new triple $T_{U,2}: P_{U,2} \longrightarrow J_{n+1}$.

Continue repeating the above alterations using the "new" triple each time until, after say q times, we have $P^*_{U,q} = M$.

(5.14) Finally, let $R'_{n+1} = T^{-1}_{U,q}$ and $K'_{n+1} = P_{U,q}$. We leave to the reader to verify that the triple $R'_{n+1}: J_{n+1} \longrightarrow K'_{n+1}$ satisfies properties $(5.1.1)^{n+1}$ – $(5.1.8)^{n+1}$ except that the collection of components of sets in K'_{n+1} may not be simple and that some components of $R'_{n+1}(j_{n+1})$ may contain more than one of the m cells in $A'_{j_{n+1}}$. Property $(5.1.8)^{n+1}$ follows from properties (5.10.1) and (5.12.1) (it is also necessary to observe that, because of (5.12.3), if C is a component of $\{T_U^{-1}(j_{n+1}) | v \in j_{n+1}\}$ with $C \cap S(v) \neq \emptyset$, then C is *not* changed by the alterations made in (5.13)). Let B_4 be a subdivision of B_3 such that each element of K'_{n+1} is a union of sets in $t(B_4)$. By running "small" tubes in various of the sets $\text{int}(\text{st}(v, \beta^2 B_4))$ where v is a vertex of $\beta^1 B_4$, the collection K'_{n+1} can be altered so that the collection of components of sets in K'_{n+1} is simple (the tubes should be small enough so that conditions $(5.1.5)^{n+1}$ and $(5.1.6)^{n+1}$ still hold); at last we have our triple $R_{n+1}: J_{n+1} \longrightarrow K_{n+1}$. Let $A_{j_{n+1}}$ be a subset of $A'_{j_{n+1}}$ chosen so that each component of $R_{n+1}(j_{n+1})$ contains exactly one m cell. This completes the proof of (5.1).

(5.15) The following corollary follows from Theorem (5.0) in much the same way as Corollary (3.7.1) followed from Theorem (3.0).

(5.15.1) Corollary. *Let M^m, N^n be compact connected p.l. manifolds with $n \geqslant m \geqslant 3$, let f be an open mapping of M onto N, and let $\epsilon > 0$. Then there is a light open mapping g from M onto N with $d(f(x), g(x)) < \epsilon$ for each $x \in M$ and with each $g^{-1}(y)$ homeomorphic to a Cantor set.*

Outline of Proof. Mimic the proof of Corollary (3.7.1) in order to find a triple $T: P \longrightarrow t(L)$ satisfying the conditions of Theorem (5.0); in place of the statement "$U_\sigma = f^{-1}(\text{int}(\text{st}^2(\sigma, t(L))^*))$ as an open connected set" use the fact that "each component of U_σ is open and maps by f onto $\text{int}(\text{st}^2(\sigma, t(L))^*)$."

The following results are immediate consequences of (3.7.1), (5.15.1), and (4.0).

(5.15.2) Corollary. *Let M^m and N^n be compact, connected p.l. manifolds with $n \geqslant m \geqslant 3$. If f is a monotone mapping of M onto N, then f can be uniformly approximated by light open mappings.*

(5.15.3) Corollary. *Let M^m and N^n be as above. A mapping f from M to N is homotopic to a light open mapping of M onto N if and only if the index of $f_*(\pi_1(M))$ in $\pi_1(N)$ is finite.*

APPENDIX. The following somewhat peculiar result is needed in the proof of (5.1); in fact, it is the "key step" in extending the technique used in [24] to construct open mappings to a technique to construct *light* open mappings. The reader is referred to [6] for the results from p.l. topology used in the proof.

(A) PROPOSITION. *Let M^m be a compact, connected p.l. manifold with triangulation L, let B be a subcomplex of L p.l. homeomorphic to an n ball, and let q be an integer larger than* 4. *Let $v_1, \ldots, v_s$ be vertices of $\beta^{q-3}L$ with each set* $\mathrm{st}(v_i, \beta^{q-2}L)^*$ *a subset of* $\mathrm{int}(B)$, $i = 1, \ldots, s$. *Then*

$$\mathrm{int}(B) - \mathrm{int}\left(\bigcup_{i=1}^{s} \mathrm{st}(\mathrm{st}(v_i, \beta^{q-2}L)^*, \beta^q L)^*\right)$$

is $m - s - 1$ connected.

PROOF. Observe that there is no constraint on how B meets ∂M and recall that $\mathrm{int}(B)$ is the *topological* interior. Let $K = \{\langle v_{i_1}, \ldots, v_{i_r}\rangle \in \beta^{q-3}L \mid \{v_{i_1}, \ldots, v_{i_r}\} \subseteq \{v_1, \ldots, v_s\}$ and $\bigcap_{j=1}^{r} \mathrm{st}(v_{i_j}, \beta^{q-2}L)^* \neq \emptyset\}$. Then K is a full subcomplex of $\beta^{q-3}L$ and $\mathrm{int}(B) - \mathrm{int}(\bigcup_{i=1}^{s} \mathrm{st}(v_i, \beta^{q-2}L)^*)$ is a strong deformation retract of $\mathrm{int}(B) - K$. Since $\bigcup_{i=1}^{r} \mathrm{st}(\mathrm{st}(v_i, \beta^{q-2}L)^*, \beta^q L)^*$ is a regular neighborhood of $\bigcup_{i=1}^{s} \mathrm{st}(v_i, \beta^{q-2}L)^*$, the

$$\mathrm{int}(B) - \mathrm{int}\left(\bigcup_{i=1}^{s} \mathrm{st}(\mathrm{st}(v_i, \beta^{q-2}L)^*, \beta^q L)^*\right)$$

is a strong deformation retract of $\mathrm{int}(B) - K$. Finally, observe that the dimension of K is at most $s - 1$ so that the result follows by general positioning.

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